

CLASS of EINSTEIN-MAXWELL DILATONS, AN ANSATZ FOR NEW FAMILIES OF ROTATING SOLUTIONS

Tonatiuh Matos^a, Darío Núñez^b, Maribel Rios^c

^aDepartamento de Física,
Centro de Investigación y de Estudios Avanzados del I. P. N.,
A. P. 14-700, 07000 México, D.F., MEXICO

^bInstituto de Ciencias Nucleares,
Universidad Nacional Autónoma de México
A. P. 70-543, 04510 México, D. F., MEXICO

^c Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Apdo. Postal 2-82, 58040 Morelia, Michoacán, México

February 7, 2008

Abstract

The functional potential formalism is used to analyze stationary axisymmetric spaces in the Einstein-Maxwell-Dilaton theory. Performing a Legendre transformation, a “Hamiltonian” is obtained, which allows to rewrite the dynamical equations in terms of three complex functions only. Using an ansatz resembling the one used by the harmonic maps ansatz, we express these three functions in terms of the harmonic parameters, studying the cases where these parameters are real, and when they are complex. For each case, the set of equations in terms of these harmonic parameters is derived, and several classes of

solutions to the Einstein-Maxwell with arbitrary coupling constant to a dilaton field are presented. Most of the known solutions of charged and dilatonic black holes are contained as special cases and can be non-trivially generalized in different ways.

PACS No. 04.20.-q, 04.20.Fy

1 Introduction

Scalar fields as a fundamental interaction in physics are one of the main predictions of the Kaluza-Klein and the Superstring theories. Scalar fields are also a fundamental component of the Brans-Dicke theory and of the inflationary models. Furthermore, in the standard model of Weinberg, Glashow and Salam scalar fields are needed as a primordial component for given mass to the particles. More recently, using an exact solution of the scalar-tensor field equations of gravity, it has been able to show that scalar fields are a very good candidate to be the dark matter in spiral galaxies [1, 2]. However, it is fair to say that there are still some open questions in this issue, which should be solved in order to give a more solid evidence of their presence, and thus establish a proof for the existence of scalar fields in nature.

The possibility of existence of scalar fields together with the spontaneous scalarization in compact stars [3] implies that astrophysical objects could contain scalar fields inherent in them. In other words, if scalar fields exist, the way it was established in [1, 2], a compact star will prefer to have one, in order to save energy. Even when these fundamental scalar fields have not been observed, they are one of the main ingredients of modern physics. Of course the question arise, why being them so important in physics have we never seen one? The answer to this question could be because they interact very weakly with matter. It can be shown that many of the theories containing scalar fields are in concordance with measurements in weak gravitational fields [4, 5]. We expect that scalar fields are important in strong gravitational fields like at the origin of the universe or in pulsars or black holes. A great effort has been done in order to detect scalar fields in strong gravitational fields [6]. In some sense, inflation at the origin of the universe could be a proof of such an interaction. Nevertheless, this effort has been done using

perturbative methods [4] or using static exact solutions [7, 8, 5]. The problem with these two approaches is that for the first, perturbative methods are very efficient only in weak fields and for the second, astrophysical objects are in general non-static, thus the approach is not realistic. Even for binary pulsars, perturbative methods have shown a great success because the distance between the pulsars is so that the gravitational field is too strong to be understood with Newtonian mechanics but enough weak to be described by perturbative methods in general relativity [6]. Nevertheless the gravitational interaction is too weak for deciding which theory containing scalar fields could be the right one. It has been possible to discard a series of theories which did not agree with measurements or to bound some parameters of some other theories using perturbative methods or static exact solutions [5, 4]. But the most interesting effects of scalar fields are expected to be very near of a black hole or of a pulsar and are expected to be non-perturbative. If we want to understand scalar fields in a strong regime one way to follow is to find rotating exact solutions of the theory containing scalar fields and comparing them with observations. The problem then is that the field equations are very complicated to be solved in an exact manner. In a past work [7] we gave a very powerful method for finding exact static solutions of the Einstein-Maxwell-Dilaton field equations using harmonic maps, we found classes of solutions with arbitrary electromagnetic fields and gravitational arbitrary multipole momentums. In the present work we want: i) to give the details of the calculations made in [7]; ii) to complete the schema of that work, and iii) to present a way to derive exact rotating dilatons with arbitrary coupling between the scalar field and the electromagnetic one. In order to do so, let us start from the Lagrangian

$$\mathcal{L} = \sqrt{-g} (R - 2(\nabla \phi)^2 - e^{-2\alpha\phi} F^2), \quad (1)$$

where g is the determinant of the metric tensor, R is the scalar curvature, ϕ the dilaton field and F the Maxwell one. The constant α is a free parameter which governs the strength of the coupling of the dilaton to the Maxwell field. When $\alpha = 0$, the action reduces to the Einstein-Maxwell scalar theory. When $\alpha = 1$, the action is part of the low-energy action of string theory. For $\alpha = \sqrt{3}$, the Lagrangian (1) leads to the Kaluza-Klein field equations obtained from the dimensional reduction of the five-dimensional Einstein vacuum equations. However, we will consider this theory for all values of α .

On the other hand, the harmonic maps ansatz has proved to be an excellent tool for finding exact solutions of systems of non-linear partial differential equations [9], in particular, this method has been very useful in solving the chiral equations derived from a non-linear σ model [10]. Einstein equations in vacuum can be reduced to a non-linear σ model with structural group $SL(2, R)$ in the space-time and to a structural group $SU(1, 1)$ in the potential spaces, *i.e.*, in terms of the Ernst potentials. The electro-vacuum case can be also reduced to a non-linear σ model with structural group $SU(2, 1)$ in terms of the extended Ernst potentials [11], [12]. The Kaluza-Klein field equations can be cast into a $SL(3, R)$ non-linear σ model in the space-time as well as in the potential space [13], [9]. This is possible because the corresponding potential space, defined below, is a symmetric Riemannian space only for $\alpha = 0$ and $\alpha = \sqrt{3}$, but this is not the case for the low energy limit in super strings theory, where $\alpha = 1$. In this work we will extend the techniques of the harmonic maps ansatz [14], [9], [15], even for non-symmetric Riemannian spaces, maintaining α as an arbitrary constant. In the present work, the reduction of the field equations to a non-linear σ model is not needed. All through this work, we will be using the MTW [16] signs conventions.

We will analyze spacetimes characterized by two Killing vector fields X and Y and introduce coordinates t and φ which are chosen such that $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial \varphi}$. The corresponding line element can then be expressed as

$$ds^2 = -f(dt - \omega d\varphi)^2 + f^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \quad (2)$$

where f , ω , and k are functions of ρ and z only. The electromagnetic potential has the form $A_\mu = (A_0, 0, 0, A_3)$, and again A_0, A_3 , and the dilatonic field, ϕ are functions of ρ and z only.

The work is composed as follows: In section 2 we introduce the abstract potential space, obtain a Lagrangian whose variation with respect to the potentials reproduce the Field equations. Performing a Legendre transformation we obtain a “Hamiltonian” and finally introduce new functions which greatly simplify the system of equations. In section 3, by means of an ansatz resembling the one of the harmonic maps, we express that system of equations in terms of harmonic parameters, analyzing two cases: where there are two real harmonic parameters and where there is one complex. In section 4 we present several new classes of solutions to the Einstein-Maxwell-Dilaton theory. Finally, in section 5 we present our conclusions and mention some possibly future developments.

2 Functional space formulation

Trying to solve directly the field equations for the Einstein-Maxwell-Dilaton theory for the line element (2) can be a very difficult task. Instead, we will apply the functional geodesic formulation by defining an abstract space whose coordinates are defined by the metric functions and the fields entering in the system. In order to introduce an ansatz resembling the harmonic map ansatz into the functional geodesic formulation (see [12], and [17] for an explanation) , we shortly explain the general idea of the harmonic map ansatz method. The field equations of the theory can be written as

$$(\rho\Psi^A_{,\varsigma})_{,\bar{\varsigma}} + (\rho\Psi^A_{,\bar{\varsigma}})_{,\varsigma} + 2\rho\{^A_{B\ C}\}\Psi^B_{,\varsigma}\Psi^C_{,\bar{\varsigma}} = 0 \quad (3)$$

where $\varsigma = \rho + i z$ and $\bar{\varsigma}$ is its complex conjugated. Ψ^A are the potentials of the geodesic formulation and $\{^A_{B\ C}\}$ are the Christoffel symbols of the Riemannian space V^A defining the potential space of the theory. Now we look for invariant transformations of the equations (3), *i.e.*, transformations of the form $\Psi^A = \Psi^A(\lambda^i)$ that leave the field equations (3) invariant, where λ^i are potentials fulfilling the same field equations (3). The potentials λ^i define the Riemannian space V_p . In terms of the potentials λ^i , the field equations (3) read

$$2\rho[\Psi^A_{,ij} + \{^A_{B\ C}\}\Psi^B_{,i}\Psi^C_{,j}]\lambda^i_{,\varsigma}\lambda^j_{,\bar{\varsigma}} + \Psi^A_{,i}[(\rho\lambda^i_{,\varsigma})_{,\bar{\varsigma}} + (\rho\lambda^i_{,\bar{\varsigma}})_{,\varsigma}] = 0, \quad (4)$$

where $,i = \partial/\partial\lambda^i$. In terms of the Christoffel symbols of the abstract Riemannian space V_p , (4) reads

$$\Psi^A_{,i;j} + \{^A_{B\ C}\}\Psi^B_{,i}\Psi^C_{,j} = 0 \quad (5)$$

where we have used the field equations for the λ^i 's and the fact that the λ^i 's are linear independent.

In what follows we will introduce the functional geodesic formulation for Lagrangian (1). The method is fully explained in [17]. Essentially, the formulation takes advantage of the fact that the introduction of a line element in the Lagrangian for the Einstein-Hilbert action, Eq. (1), and performing the variation, are operations which commute for some cases, in particular for the axisymmetric stationary one. Thus, introducing the operator $D = (\partial_\rho, \partial_z)$,

taking out a total divergence term and eliminating the terms with Dk by means of a Legendre transformation, we obtain that the original Lagrangian, given by Eq. (1), can be rewritten as:

$$\mathcal{L} = \frac{\rho}{2f^2} Df^2 - \frac{f^2}{2\rho} D\omega^2 + \frac{2\rho}{\alpha^2 \kappa^2} D\kappa^2 + \frac{2f\kappa^2}{\rho} [(\omega DA_0 + DA_3)^2 - \frac{\rho^2}{f^2} DA_0^2], \quad (6)$$

where $\kappa^2 = e^{-2\alpha\phi}$.

The Euler-Lagrange equations, obtained directly from extremizing the action for such Lagrangian: $D(\frac{\partial \mathcal{L}}{\partial D Z^a}) - (\frac{\partial \mathcal{L}}{\partial Z^a}) = 0$, with $Z^a = (f, \omega, A_0, A_3, \kappa)$, are:

the Klein-Gordon equation:

$$D^2 \kappa + \left(\frac{D\rho}{\rho} - \frac{D\kappa}{\kappa} \right) D\kappa - \frac{\alpha^2 \kappa^3 f}{\rho^2} [(\omega DA_0 + DA_3)^2 - \frac{\rho^2}{f^2} DA_0^2] = 0, \quad (7)$$

the Maxwell equations:

$$\begin{aligned} D \left(\frac{f\kappa^2}{\rho} (\omega DA_0 + DA_3) \right) &= 0, \\ D \left(\kappa^2 \left[\frac{f\omega}{\rho} (\omega DA_0 + DA_3) - \frac{\rho}{f} DA_0 \right] \right) &= 0, \end{aligned} \quad (8)$$

and the main Einstein's equations:

$$\begin{aligned} D^2 f + \left(\frac{D\rho}{\rho} - \frac{Df}{f} \right) Df + \frac{f^3}{\rho^2} D\omega^2 - \frac{2\kappa^2 f^2}{\rho^2} [(\omega DA_0 + DA_3)^2 + \frac{\rho^2}{f^2} DA_0^2] &= 0, \\ D^2 \omega - \left(\frac{D\rho}{\rho} - \frac{2Df}{f} \right) D\omega + \frac{4\kappa^2}{f} (\omega DA_0 + DA_3) DA_0 &= 0 \end{aligned} \quad (9)$$

which are equivalent to the usual field equations. Notice that the metric function k does not appear, reflecting the fact that it is determined by quadratures in terms of the rest of the functions. The Lagrangian (6) sometimes is viewed as describing the line element in the potential space, and thus the motion equations can be seen as geodesics in that space.

From the fact that $D\tilde{D} = 0$ for any analytic function, with $\tilde{D} = (-\partial_z, \partial_\rho)$, we conclude from the second Maxwell equation, Eq.(8), that there exists a potential χ , such that

$$\tilde{D}\chi = \frac{2f\kappa^2}{\rho} (\omega DA_0 + DA_3). \quad (10)$$

And with this potential, the second Einstein's equation, Eq. (9), can be rewritten as $D(\frac{f^2}{\rho} D\omega + \psi \tilde{D}\chi) = 0$, with $\psi = 2A_0$, so that there exists another potential, ϵ , defined by

$$\tilde{D}\epsilon = \frac{f^2}{\rho} D\omega + \psi \tilde{D}\chi. \quad (11)$$

The use of these potentials χ and ϵ , will be helpful in the procedure of defining harmonic functions, so we rewrite the field equations in terms of these functions:

$$\begin{aligned} D^2\kappa + \left(\frac{D\rho}{\rho} - \frac{D\kappa}{\kappa}\right) D\kappa + \frac{\alpha^2\kappa^3}{4f}(D\psi^2 - \frac{1}{\kappa^4} D\chi^2) &= 0, \\ D^2\psi + \left(\frac{D\rho}{\rho} + \frac{2D\kappa}{\kappa} - \frac{Df}{f}\right) D\psi - \frac{1}{\kappa^2 f}(D\epsilon - \psi D\chi) D\chi &= 0, \\ D^2\chi + \left(\frac{D\rho}{\rho} - \frac{2D\kappa}{\kappa} - \frac{Df}{f}\right) D\chi + \frac{\kappa^2}{f}(D\epsilon - \psi D\chi) D\psi &= 0, \\ D^2f + \left(\frac{D\rho}{\rho} - \frac{Df}{f}\right) Df + \frac{1}{f}(D\epsilon - \psi D\chi)^2 - \frac{\kappa^2}{2}(D\psi^2 + \frac{1}{\kappa^4} D\chi^2) &= 0, \\ D^2\epsilon - D\psi D\chi - \psi D^2\chi + \left(\frac{D\rho}{\rho} - \frac{2Df}{f}\right)(D\epsilon - \psi D\chi) &= (12) \end{aligned}$$

where we have used the fact that $\tilde{D}A\tilde{D}B = DA\tilde{D}B$, for any functions A, B . The equation for χ is obtained from $D\tilde{D}A_3 = 0$, and the one for ϵ is obtained from $D\tilde{D}\omega = 0$. Equations (12) are equivalent to equations (3). This new set of field equations can be obtained from the Lagrangian [11], [14]

$$\mathcal{L} = \frac{\rho}{2f^2} [Df^2 + (D\epsilon - \psi D\chi)^2] + \frac{2\rho}{\alpha^2\kappa^2} D\kappa^2 - \frac{\rho\kappa^2}{2f} (D\psi^2 + \frac{1}{\kappa^4} D\chi^2). \quad (13)$$

Notice, however, that this new Lagrangian is not obtained if the transformation is made directly on the original Lagrangian, given by Eq.(1) or Eq.(6). This fact implies that the transformations defined by Eq.(10), and Eq.(11) must have a degeneracy. A detailed analysis of this issue is under work and will be published elsewhere.

As mentioned above, this Lagrangian, Eq.(13), can be seen as given by the line element, DS^2 , of a potential space: $\mathcal{L} = DS^2 = G_{AB} D\Psi^A D\Psi^B$, with $\Psi^A = (f, \epsilon, \chi, \psi, \kappa)$, so that the equations of motion, Eqs.(12), obtained

from variations of this Lagrangian with respect to the “coordinates”, Ψ^A , can be thought as “geodesics in such potential space”.

It is important to recall some of the geometric properties of this potential space, DS^2 . It defines a Riemannian space with constant scalar curvature, $R = -(12 + \alpha^2)$. All the covariant derivatives of the Riemann tensor are proportional to $\alpha^2 - 3$, thus the corresponding Riemannian space is symmetric only for the case $\alpha^2 = 3$ (and for the case $\alpha = 0$, not treated here). Therefore, for an arbitrary α , the space is not symmetric and thus the harmonic map ansatz formulation can not be applied.

In this way, we see that in order to reformulate the Einstein-Maxwell-Dilaton field equations for arbitrary α , including the low energy limit of string theory, we cannot use the harmonic map formulation, but we can try instead to mimic that formulation, by means of appropriately chosen ansatz. In this quest, it is necessary to obtain the algebraic structure associated with the potential space. An analysis of the killing vectors, ξ^a , implies that there are 8 of them for the space describe by Eq.(13):

$$\begin{aligned}
\xi_1^a &= a_1 (0, 1, 0, 0, 0), \\
\xi_2^a &= a_2 (0, 0, 1, 0, 0), \\
\xi_3^a &= a_3 (f, \epsilon, \frac{\chi}{2}, \frac{\psi}{2}, 0), \\
\xi_4^a &= a_4 (0, \chi, 0, 1, 0) \\
\xi_5^a &= a_5 (0, 0, \chi, -\psi, \kappa) \\
\xi_6^a &= a_6 [(2\epsilon - \chi\psi) f, \epsilon^2 - f^2 + f\psi^2\kappa^2, \\
&\quad f\psi\kappa^2 + \epsilon\chi, \psi\epsilon - \psi^2\chi - \frac{f\chi}{\kappa^2}, \frac{\alpha^2\psi\chi\kappa}{2}], \\
\xi_7^a &= a_7 [f\chi, f\psi\kappa^2 + \epsilon\chi, \\
&\quad f\kappa^2 + (1 + \alpha^2)\frac{\chi^2}{4}, \epsilon + (1 - \alpha^2)\frac{\psi\chi}{2}, \alpha^2\frac{\chi\kappa}{2}], \\
\xi_8^a &= a_8 [f\psi, (3 - \alpha^2)\frac{\chi\psi^2}{4}, (3 - \alpha^2)\frac{\chi\psi}{2} - \epsilon, \\
&\quad \frac{f}{\kappa^2} + (1 + \alpha^2)\frac{\psi^2}{4}, -\alpha^2\frac{\psi\kappa}{2}],
\end{aligned} \tag{14}$$

where a_1, \dots, a_8 , are arbitrary constants. For $\alpha^2 \neq 3$, only the first 5 killing vectors remain independent. Thus, the algebra associated with the potential

space for arbitrary values of α is $SL(3, R)$ for $\alpha^2 = 3$, and a subalgebra of $SL(3, R)$, the upper triangle one in the matrix representation, for $\alpha^2 \neq 3$.

The commutation relations, $[\xi_i^a, \xi_j^b] = \xi_{i;b}^a \xi_j^b - \xi_{j;b}^a \xi_i^b$ for the first five killing vectors are:

$$\begin{aligned} [\xi_1, \xi_2] &= 0, \quad [\xi_1, \xi_3] = -\xi_1, \quad [\xi_1, \xi_4] = 0, \quad [\xi_1, \xi_5] = 0, \\ [\xi_2, \xi_3] &= -\frac{1}{2}\xi_2, \quad [\xi_2, \xi_4] = -\xi_1, \quad [\xi_2, \xi_5] = -\xi_2, \\ [\xi_3, \xi_4] &= -\frac{1}{2}\xi_4, \quad [\xi_3, \xi_5] = 0, \quad [\xi_4, \xi_5] = \xi_4. \end{aligned} \quad (15)$$

Notice that we have several subalgebras with three vectors, but all of them have at least one of the commutators equal to zero. These results will be used in studying the type of target space that can be chosen in doing a map to a bi-dimensional space.

Finally, as we mentioned, the remaining metric function k is determined by quadratures in terms of the other field functions [7], explicitly:

$$\begin{aligned} k_\rho &= \frac{\rho}{4f^2} [f_\rho^2 - f_z^2 + \epsilon_\rho^2 - \epsilon_z^2 + (\psi^2 + \frac{f}{\kappa^2})(\chi_\rho^2 - \chi_z^2) - 2\psi(\epsilon_\rho \chi_\rho - \epsilon_z \chi_z) \\ &\quad + f\kappa^2(\psi_\rho^2 - \psi_z^2) + (\frac{2f}{\alpha\kappa})^2(\kappa_\rho^2 - \kappa_z^2)], \end{aligned} \quad (17)$$

$$k_z = \frac{\rho}{2f^2} [f_\rho f_z + \epsilon_\rho \epsilon_z + \kappa^2 f \psi_\rho \psi_z - \psi(\epsilon_\rho \chi_z + \epsilon_z \chi_\rho) + \quad (18)$$

$$(\psi^2 + \frac{f}{\kappa^2}) \chi_\rho \chi_z + (\frac{2f}{\alpha\kappa})^2 \kappa_\rho \kappa_z]. \quad (19)$$

Continuing with the quest for reformulating the Lagrangian, Eq. (13), notice that we do can perform a Legendre transformation and defining “momenta”, as $P_a = \frac{\partial \mathcal{L}}{\partial D\Psi^a}$, we can construct a new function which is along the lines of the standard Hamiltonian although in our case, it has not the usual properties of evolution associated with the Hamiltonians. This “Hamiltonian” has the explicit form:

$$\mathcal{H} = \frac{f^2}{2\rho} (P_f^2 + P_\epsilon^2) + \frac{\alpha^2 \kappa^2}{8\rho} P_\kappa^2 - \frac{f}{2\rho} [\frac{P_\psi^2}{\kappa^2} + \kappa^2 (P_\chi + \psi P_\epsilon)^2]. \quad (20)$$

The equations of motion are now, $D\Psi^a = \frac{\partial \mathcal{H}}{\partial P_a}$, and $D P_a = -\frac{\partial \mathcal{H}}{\partial \Psi^a}$, that is:

$$D f = \frac{f^2}{\rho} P_f,$$

$$\begin{aligned}
D\epsilon &= \frac{f^2}{\rho} \left[P_\epsilon - \frac{\kappa^2 \psi}{f} (P_\chi + \psi P_\epsilon) \right], \\
D\psi &= -\frac{f}{\rho \kappa^2} P_\psi, \\
D\chi &= -\frac{f \kappa^2}{\rho} (P_\chi + \psi P_\epsilon), \\
D\kappa &= \frac{\alpha^2 \kappa^2}{4\rho} P_\kappa,
\end{aligned}$$

and

$$\begin{aligned}
DP_f &= -\frac{f}{\rho} (P_f^2 + P_\epsilon^2) + \frac{1}{2\rho} \left[\frac{P_\psi^2}{\kappa^2} + \kappa^2 (P_\chi + \psi P_\epsilon)^2 \right], \\
DP_\epsilon &= 0, \\
DP_\psi &= \frac{f \kappa^2}{\rho} (P_\chi + \psi P_\epsilon) P_\epsilon, \\
DP_\chi &= 0, \\
DP_\kappa &= -\frac{f}{\rho \kappa} \left[\frac{P_\psi^2}{\kappa^2} - \kappa^2 (P_\chi + \psi P_\epsilon)^2 \right] - \frac{\alpha^2 \kappa}{4\rho} P_\kappa^2. \tag{21}
\end{aligned}$$

Now we can define new functions in order to simplify the form of this last set of equations, and to be able to introduce the harmonic maps:

$$\begin{aligned}
A &= \frac{f}{2\rho} (P_f - i P_\epsilon), \\
B &= \frac{\sqrt{f}}{2\rho} \left[\frac{P_\psi}{\kappa} - i \kappa (P_\chi + \psi P_\epsilon) \right], \\
C &= \frac{\alpha^2 \kappa}{4\rho} P_\kappa,
\end{aligned}$$

or in terms of the “velocities”:

$$\begin{aligned}
A &= \frac{1}{2f} [Df - i (D\epsilon - \psi D\chi)], \\
B &= -\frac{1}{2\sqrt{f}} (\kappa D\psi - \frac{i}{\kappa} D\chi), \\
C &= \frac{D\kappa}{\kappa}
\end{aligned}$$

so we can rewrite the equations for the “momenta”, Eq.(21) as the following system:

$$\begin{aligned}\frac{1}{\rho}D(\rho A) &= A(A - A^*) + B B^*, \\ D(\rho B) &= -\frac{1}{2}B(A - 3A^*) - C B^*, \\ D(\rho C) &= -\frac{\alpha^2}{2}(B^2 + B^{*2}),\end{aligned}\tag{22}$$

where $*$ denotes complex conjugate. Notice that in this way, we have reduced the system of field equations to a set of three first order differential equations for the three functions A, B , and C .

3 The Ansatz

Now we introduce the harmonic maps ansatz. Let V_p be a p dimensional Riemannian space, and λ^i an harmonic parameters in V_p , *i.e.*

$$(\rho\lambda_{,\zeta}^i)_{,\bar{\zeta}} + (\rho\lambda_{,\bar{\zeta}}^i)_{,\zeta} + 2\rho\Gamma_{jk}^i\lambda_{,\zeta}^j\lambda_{,\bar{\zeta}}^k = 0,\tag{23}$$

where Γ_{jk}^i are the Christoffel symbols on the Riemannian space V_p . We suppose that all the potentials f, ϵ, \dots depend on the λ^i 's, $f = f(\lambda^i)$, $\epsilon = \epsilon(\lambda^i), \dots$ etc. (for details see ref. [12] [9] and [10]). We want to recall that this work belongs to a program aimed to obtain exact solutions of the field equations derived from the Lagrangian (6) by means of the harmonic maps ansatz (see [9]) for the case of arbitrary α . However, as we mentioned, some difficulties appear for arbitrary α and we cannot follow the procedures carried out before. First, as was shown above, for $\alpha = 0, \sqrt{3}$, the isometry group has eight parameters, whereas for arbitrary $\alpha(\neq 0, \sqrt{3})$, there exist only five parameters [11]. Second, the isometry group of this Lagrangian is trivial except for $\alpha = 0$ and $\alpha = \sqrt{3}$, in the sense that the invariant transformations of this Lagrangian for arbitrary α lead only to gauge transformations. Third, since for arbitrary $\alpha(\neq 0, \sqrt{3})$ the potential space is not symmetric, it is not possible to obtain a no-linear sigma-model for this system. All these problems prevent us to take the same way as in previous works. Therefore

we propose another method for solving the differential equations. Here we will study the 2-dimensional subspace of the V_p -space. We start with a two dimensional Riemannian spaces V_2 with constant curvature, parameterizing this Riemannian spaces with two harmonic parameters λ , and τ , such that $\lambda, \tau \in \mathbb{R}$. The line element is

$$ds^2 = \frac{2(d\lambda^2 + d\tau^2)}{(1 - \sigma(\lambda^2 + \tau^2))^2} = \frac{d\xi d\xi^*}{(1 - \sigma\xi\xi^*)^2}, \quad (24)$$

where σ is a real constant proportional to the potential space curvature, and $\xi = \lambda + i\tau$, for the case of complex parameters. We know that this is a maximally symmetric space, so it has three killing vectors. If the electromagnetic field vanishes any value for α is similar, because there is not interaction between scalar and electromagnetic fields. Then σ can be set to one as for $\alpha^2 = 0, 3$. But if there is electromagnetic interaction the situation is different. As we showed, the subalgebras for the potential space with arbitrary α , with three Killing vectors, are such that one of them has to be set to zero, so we conclude that if the electromagnetic field does not vanish the only case of maximally symmetric V_2 that can be taken is the one with $\sigma = 0$. In this case, the parameters satisfy the usual Laplace equation: $D(\rho D\lambda) = 0$, $D(\rho D\tau) = 0$. As in the harmonic maps ansatz case, let us express the functions A, B, C in terms of these parameters as follows:

$$\begin{aligned} A &= a_1(\lambda, \tau) D\lambda + a_2(\lambda, \tau) D\tau, \\ B &= b_1(\lambda, \tau) D\lambda + b_2(\lambda, \tau) D\tau, \\ C &= c_1(\lambda, \tau) D\lambda + c_2(\lambda, \tau) D\tau. \end{aligned} \quad (25)$$

Using the harmonic equations, *i.e.* the Laplace equation, for these parameters in the field equations (22), and recalling the fact that $(D\lambda)^2$, $(D\tau)^2$, and $D\lambda D\tau$ are independent functions, from the system of equations for A, B, C , Eq.(22), we obtain the following set of equations:

$$\begin{aligned} a_{1,\lambda} - a_1(a_1 - a_1^*) - b_1 b_1^* &= 0, \\ b_{2,\lambda} + \frac{b_1}{2}(a_1 - 3a_1^*) + c_1 b_1^* &= 0, \\ c_{1,\lambda} + \frac{\alpha^2}{2}(b_1^2 + b_1^{*2}) &= 0, \end{aligned} \quad (26)$$

$$\begin{aligned}
a_{2,\tau} - a_2 (a_2 - a_2^*) - b_2 b_2^* &= 0, \\
b_{2,\tau} + \frac{b_2}{2} (a_2 - 3 a_2^*) + c_2 b_2^* &= 0, \\
c_{2,\tau} + \frac{\alpha^2}{2} (b_2^2 + b_2^{*2}) &= 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
a_{1,\tau} + a_{2,\lambda} - 2 a_1 a_2 + a_1^* a_2 + a_1 a_2^* - b_1 b_2^* - b_1^* b_2 &= 0, \\
2 b_{1,\tau} + 2 b_{2,\lambda} + b_2 (a_1 - 3 a_1^*) + b_1 (a_2 - 3 a_2^*) + 2 c_1 b_2^* + 2 c_2 b_1^* &= 0, \\
c_{1,\tau} + c_{2,\lambda} + \alpha^2 (b_1 b_2 + b_1^* b_2^*) &= 0.
\end{aligned} \tag{28}$$

Equations (26, 27, 28) are equivalent to the field equations (4). Taking the original potential also as functions of the harmonic parameters, we can express the functions A, B , and C from Eq.(25) as:

$$\begin{aligned}
A &= \frac{1}{2f} [f_{,\lambda} - i (\epsilon_{,\lambda} - \psi \chi_{,\lambda})] D \lambda + \frac{1}{2f} [f_{,\tau} - i (\epsilon_{,\tau} - \psi \chi_{,\tau})] D \tau, \\
B &= -\frac{1}{2\sqrt{f}} (\kappa \psi_{,\lambda} - \frac{i}{\kappa} \chi_{,\lambda}) D \lambda - \frac{1}{2\sqrt{f}} (\kappa \psi_{,\tau} - \frac{i}{\kappa} \chi_{,\tau}) D \tau, \\
C &= \frac{\kappa_{,\lambda}}{\kappa} D \lambda + \frac{\kappa_{,\tau}}{\kappa} D \tau,
\end{aligned} \tag{29}$$

from which, and from Eq.(25), we can make the following identification:

$$\begin{aligned}
a_1 &= \frac{1}{2f} [f_{,\lambda} - i (\epsilon_{,\lambda} - \psi \chi_{,\lambda})]; & a_2 &= \frac{1}{2f} [f_{,\tau} - i (\epsilon_{,\tau} - \psi \chi_{,\tau})], \\
b_1 &= -\frac{1}{2\sqrt{f}} (\kappa \psi_{,\lambda} - \frac{i}{\kappa} \chi_{,\lambda}); & b_2 &= -\frac{1}{2\sqrt{f}} (\kappa \psi_{,\tau} - \frac{i}{\kappa} \chi_{,\tau}), \\
c_1 &= \frac{\kappa_{,\lambda}}{\kappa}; & c_2 &= \frac{\kappa_{,\tau}}{\kappa}.
\end{aligned} \tag{30}$$

4 Classes of solutions

Now we proceed to present some solutions to the Einstein-Maxwell-Dilaton system in terms of the harmonic functions λ, τ . Taking a_1, b_1, c_1 as functions of λ only, and a_2, b_2, c_2 as functions of τ only, we obtain two sets of equations, one in terms of λ , the other in terms of τ , with the equations Eqs.(28), being only constraint equations. We will see three such cases:

4.1 First Class

Letting all the functions be complex, we obtain that $a_2 = b_2 = c_2 = 0$, and

$$\begin{aligned} a_1 &= -\frac{1 + i\sqrt{\gamma^2 - \frac{1}{4}}}{2(\lambda + \beta_0)}, \\ b_1 &= -\frac{\sqrt{\gamma}(\sqrt{\gamma - \frac{1}{2}} + i\sqrt{\gamma + \frac{1}{2}})}{\alpha(\lambda + \beta_0)}, \\ c_1 &= -\frac{\gamma}{\lambda + \beta_0}, \end{aligned} \quad (31)$$

where β_0 is an arbitrary constant and $\gamma = \frac{\alpha}{2} \sqrt{\frac{3}{4-\alpha^2}}$. This solution is valid for $1 < \alpha < 2$. Using Eqs.(30), we obtain that the potentials are given by:

$$\begin{aligned} f &= \frac{f_0}{\lambda + \beta_0}, \\ \epsilon &= \frac{\epsilon_0}{\lambda + \beta_0} + \epsilon_{10}(\lambda + \beta_0)^{-(\gamma + \frac{1}{2})}, \\ \chi &= \frac{2\kappa_0}{\alpha} \sqrt{\frac{f_0\gamma}{\gamma + \frac{1}{2}}} [(\lambda + \beta_0)^{-(\gamma + \frac{1}{2})} - \beta_0^{-(\gamma + \frac{1}{2})}], \\ \psi &= \frac{2}{\kappa_0\alpha} \sqrt{\frac{f_0\gamma}{\gamma - \frac{1}{2}}} [(\lambda + \beta_0)^{\gamma - \frac{1}{2}} - \beta_0^{\gamma - \frac{1}{2}}], \\ \kappa &= \kappa_0(\lambda + \beta_0)^{-\gamma}, \end{aligned} \quad (32)$$

where $\epsilon_0 = f_0 \sqrt{\frac{\gamma + \frac{1}{2}}{\gamma - \frac{1}{2}}} (\frac{4-\alpha^2}{\alpha^2} \gamma + \frac{1}{2})$, and $\epsilon_{10} = -\frac{4f_0\gamma\beta_0^{\gamma - \frac{1}{2}}}{\alpha^2 \sqrt{\gamma^2 - \frac{1}{4}}}$. From this potentials, using Eqs.(11, 10), we obtain that

$$\begin{aligned} \omega_{,\rho} &= -\frac{\sqrt{\gamma^2 - \frac{1}{4}}}{f_0} \rho \lambda_{,z}, \quad \omega_{,z} = \frac{\sqrt{\gamma^2 - \frac{1}{4}}}{f_0} \rho \lambda_{,\rho}, \\ A_{3,\rho} &= -\frac{\sqrt{\gamma f_0}}{\kappa_0 \alpha} (\lambda + \beta_0)^{\gamma - \frac{3}{2}} (\omega \sqrt{\gamma - \frac{1}{2}} \lambda_{,\rho} - \frac{\rho}{f_0} \sqrt{\gamma + \frac{1}{2}} (\lambda + \beta_0) \lambda_{,z}), \\ A_{3,z} &= -\frac{\sqrt{\gamma f_0}}{\kappa_0 \alpha} (\lambda + \beta_0)^{\gamma - \frac{3}{2}} (\omega \sqrt{\gamma - \frac{1}{2}} \lambda_{,z} + \frac{\rho}{f_0} \sqrt{\gamma + \frac{1}{2}} (\lambda + \beta_0) \lambda_{,\rho}) \end{aligned} \quad (33)$$

Finally, from Eq.(19), we obtain that the last metric coefficient, k , in this case is constant, $k = k_0$, and we take $k_0 = 0$. This solution represents a rotating object with scalar and electromagnetic fields which depend on the form of λ . In this way, we have obtained a family of solutions for the Einstein-Maxwell-Dilaton theory in which all the fields are non-trivially involved. Let us give an example. Writing the metric (2) in spherical like coordinates $\rho = r \sin \theta$, $z = r \cos \theta$, we can choose $\lambda = M/r$, which give us the line element of the space-time as ($f_0 = \beta_0 = \frac{1}{2}$):

$$ds^2 = -\frac{1}{1 + \frac{2M}{r}}(dt - 2\sqrt{\gamma^2 - \frac{1}{4}}M \cos \theta d\phi)^2 + (1 + \frac{2M}{r})(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (34)$$

the scalar field is

$$\varphi = \ln(\beta_0 + \frac{M}{r})^{\frac{\gamma}{\alpha}} - \varphi_0, \quad (35)$$

where $\varphi_0 = \ln \kappa_0^\alpha$, and the electric and magnetic potentials are

$$A_0 = \sqrt{\frac{\gamma}{\gamma - \frac{1}{2}}} \frac{[(1 + \frac{2M}{r})^{\gamma - \frac{1}{2}} - 1]}{2^\gamma \kappa_0 \alpha}, \quad (36)$$

$$A_3 = \sqrt{\gamma(\gamma + \frac{1}{2})} \frac{M}{2^{\gamma-1} \kappa_0 \alpha} [(\beta_0 + \frac{2M}{r})^{\gamma - \frac{1}{2}} - 1] \cos \theta. \quad (37)$$

Metric (34) contains a mass-like parameter, M , and is not asymptotically flat, except for $\gamma^2 = \frac{1}{4}$. This space-time represents a multipole electromagnetic field copled to a multiple gravitational one. The electric charge is $q = \sqrt{\gamma(\gamma - \frac{1}{2})}M/(2^{\gamma-1}\kappa_0\alpha)$, and magnetic monopole charge $Q = \sqrt{\gamma(\gamma - \frac{1}{2})}M(\beta_0^{\gamma - \frac{1}{2}} - 1)/(2^{\gamma-1}\kappa_0\alpha)$. Notice, however that the scalar field is tighted to the gravitational and electromagnetic charges, in the sense that when $M = 0$, or $\gamma = 0$, the scalar field becomes trivial. In order to overcome this problem, we think that more potentials have to be included. Also, this solution has no horizons, and is singular for $r = 0$. Thus, we consider that this solution is a good one for describing the space time with gravitational, electromagnetic and dilatonic fields in a region between $r > 0$, and $r = r_0$ where it has to be matched to an exterior solution. A detailed analysis of these physical properties of this particular exact solution, as well as other solutions belonging to the family of solutions presented in this section is underwork and will be published presently.

4.2 Second Class

We take the case where the ansatz is such that $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, and that a_1, b_1, c_1 depend only on λ , and a_2, b_2, c_2 depend only on τ . With these conditions the system of equations, Eqs. (26) implies that $b_1 b_2 = 0$. Taking $b_1 = 0$, in turn implies that $a_1 = c_1 = \text{const}$, and the final set of equations reduces to the following:

$$a_{2,\tau} - b_2^2 = 0, \quad b_{2,\tau} - b_2(a_2 - c_2) = 0, \quad c_{2,\tau} + \alpha^2 b_2^2 = 0.$$

Solving for a_2 , we get a Riccati type equation:

$$a_{2,\tau\tau} - (1 + \alpha^2)(a_2^2)_{,\tau} + 2k_0 a_{2,\tau} = 0, \quad (38)$$

with k_0 an integration constant. We obtain two sets of solutions for a_2 :

$$a_2 = \frac{k_1 e^{q_1 \tau} - k_2 e^{(q_1 + 2(1 + \alpha^2)) \tau}}{k_1 e^{q_1 \tau} + k_2 e^{(q_1 + 2(1 + \alpha^2)) \tau}} + \frac{k_0}{(1 + \alpha^2)}, \quad (39)$$

and

$$a_2 = \frac{1}{(1 + \alpha^2)} \left(-\frac{k_3}{k_3 \tau + k_4} + k_0 \right), \quad (40)$$

which imply

$$\begin{aligned} b_2 &= \frac{2 \sqrt{-k_1 k_2 (1 + \alpha^2)} e^{2(q_1 + 1 + \alpha^2) \tau}}{k_1 e^{q_1 \tau} + k_2 e^{(q_1 + 2(1 + \alpha^2)) \tau}}, \\ c_2 &= \frac{k_0}{(1 + \alpha^2)} - \alpha^2 \frac{k_1 e^{q_1 \tau} - k_2 e^{(q_1 + 2(1 + \alpha^2)) \tau}}{k_1 e^{q_1 \tau} + k_2 e^{(q_1 + 2(1 + \alpha^2)) \tau}}, \end{aligned} \quad (41)$$

and

$$\begin{aligned} b_2 &= \frac{k_3}{(1 + \alpha^2)^{\frac{1}{2}} (k_3 \tau + k_4)}, \\ c_2 &= \frac{1}{(1 + \alpha^2)} \left(k_0 + \frac{\alpha^2 k_3}{k_3 \tau + k_4} \right), \end{aligned} \quad (42)$$

where k_i, q_i are constants. Performing the integration in equations (30), the following expressions for the potentials are obtained:

$$f = f_0 \frac{e^{\lambda_0 \lambda + \tau_0 \tau}}{(m_1 \Sigma_1 + m_2 \Sigma_2)^\gamma},$$

$$\begin{aligned}
\kappa^2 &= \kappa_0^2 (m_1 \Sigma_1 + m_2 \Sigma_2)^\beta e^{\lambda_0 \lambda + (\tau_0 - t_1 - t_2) \tau}, \\
\psi &= \frac{m_3 \Sigma_1 + m_4 \Sigma_2}{(m_1 \Sigma_1 + m_2 \Sigma_2)}, \\
\chi &= 0, \\
\epsilon &= 0,
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
f &= \frac{f_0 e^{\gamma k_1 \tau + \lambda_0 \lambda}}{(k_3 \tau + k_4)^\gamma}, \\
\kappa^2 &= \kappa_0^2 (k_3 \tau + k_4)^\beta e^{\gamma k_1 \tau + \lambda_0 \lambda}, \\
\psi &= -2 f_0^{1/2} \kappa_0 (1 + \alpha^2)^{1/2} (k_3 \tau + k_4), \\
\chi &= 0, \\
\epsilon &= 0,
\end{aligned} \tag{44}$$

where $\gamma = \frac{2}{1+\alpha^2}$, $\beta = \alpha^2 \gamma$, $f_0, \kappa_0, t_i, \lambda_0, \tau_0, m_i$ are constants. For the first set of solutions $\Sigma_i = e^{t_i \tau}$ and the constants are related by

$$4 m_1 m_2 f_0 + \kappa_0^2 (1 + \alpha^2) (m_1 m_4 - m_2 m_3)^2 = 0. \tag{45}$$

And for the second set of solutions $t_1 = -t_2$, $\Sigma_1 = \tau$, $\Sigma_2 = 1$, and the constants satisfy the relationship

$$4 m_1^2 f_0 - \kappa_0^2 (1 + \alpha^2) (m_1 m_4 - m_2 m_3)^2 = 0. \tag{46}$$

In this way, with the ansatz chosen, we obtained families of solutions for the Einstein-Maxwell-Dilaton theory, without magnetic field and without rotation. The generic solutions are in terms of two arbitrary harmonic functions and a large number of constants. The respectively magnetic solutions can be obtained using the invariant transformations .

$$\phi \rightarrow -\phi, \quad F_{\mu\nu} = 1/2 e^{-2\alpha\phi} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{47}$$

Thus, we can generate particular solutions with very different physical properties. These sets of solutions were presented as a rapid communication in [7], and it was shown that several interesting known solutions are included, such as the static dilatonic version of the Kastor-Traschen solution [18], and

several generalizations of it; the spherically symmetric dilatonic black hole, [19]; and solutions with arbitrary magnetic field coupled to the dilaton [20].

In order to present explicitly some families of solutions, let us now rewrite solutions (32) in a more convenient form. If we perform the following transformation, $f \rightarrow f/f_0$, $\kappa^2 \rightarrow \kappa^2/\kappa_0^2$, $\lambda \rightarrow \lambda_0\lambda + \tau\tau_0$, $\tau_0 \rightarrow -t_1 - t_2$, and $g = m_1 e^{t_1\tau} + m_2 e^{t_2\tau}$ in (43), then solution (43) transforms into

$$\begin{aligned} f &= \frac{e^\lambda}{g^\gamma}, \\ \kappa^2 &= \frac{e^{-\lambda+\tau_0\tau}}{g^\beta}, \\ \psi &= \frac{(m_3\Sigma_1 + m_4\Sigma_2)}{g}, \\ \chi &= 0 \\ \epsilon &= 0, \end{aligned} \tag{48}$$

or analogously for solution (44) we perform the transformation $f \rightarrow f/f_0$, $\kappa^2 \rightarrow \kappa^2/\kappa_0^2$, $\lambda \rightarrow \lambda_0\lambda + \gamma k_1\tau$, $\psi \rightarrow -\psi \kappa_0(1 + \alpha^2)^{1/2}/f_0^{1/2}$, and $g = k_3\tau + k_4$ then solution (44) now reads

$$\begin{aligned} f &= \frac{e^\lambda}{g^\gamma}, & \kappa^2 &= e^\lambda g^\beta, \\ \psi &= \frac{1}{g}, & \chi &= 0, \quad \epsilon = 0. \end{aligned} \tag{49}$$

In these two cases the differential equation for the metric function k in (19) can be separated into a gravitational, a scalar and a electromagnetic parts. In order to do so, we substitute (48) and (49) into (19) to obtain the differential equation for k , thus arriving at

$$k_{,\zeta} = \frac{\rho}{2}[(\lambda_{,\zeta})^2 + \frac{1}{\alpha^2}((\lambda_{,\zeta} - \tau_0\tau_{,\zeta})^2 - 2q_1q_2\beta(\tau_{,\zeta})^2)], \tag{50}$$

where $\zeta = \rho + iz$. Let us now perform the following separation

$$k = k_g + k_e + k_s, \tag{51}$$

where we have defined the gravitational part of k as

$$k_{g,\zeta} = \frac{\rho}{2}(\lambda_{,\zeta})^2, \quad (52)$$

the scalar part as

$$k_{s,\zeta} = \frac{\rho}{2\alpha^2}(\lambda_{,\zeta} - \tau_0\tau_{,\zeta})^2, \quad (53)$$

and the electromagnetic part as

$$k_{e,\zeta} = \frac{\rho}{\alpha^2}q_1q_2\beta(\tau_{,\zeta})^2, \quad (54)$$

where now the constants a_1, \dots, q_1, q_2 , and κ_0 , satisfy the relationship

$$4a_1^2 - \kappa_0^2(1 + \alpha^2)(a_1a_4 - a_2a_3)^2 = 0. \quad (55)$$

It is important to note that the electrostatic potential ψ is completely determined by the harmonic function τ . This means that the electrostatic (magnetostatic) potential is determined only by τ , so we can obtain solutions with arbitrary electromagnetic fields writing the corresponding solution of the Laplace equation for τ . The corresponding magnetic solution can be obtained using the invariant transformations (47). The most important well-known solutions can be derived from this method. Some examples are given in [21]. New solutions have been derived in [22], [23] [24].

On the other hand, a star is basically a gravitational monopole together with a magnetic dipole field. Using the invariant transformations (47), we can construct a class of solution with this characteristics. For the gravitational potential we now chose a gravitational monopole, in order to reproduce the most important gravitational features of the star and of the Schwarzschild solution. In order to do so we write the line element (2) in Boyer-Lindquist coordinates $\rho = \sqrt{r^2 - 2mr} \sin \theta$, $z = (r - m) \cos \theta$ and take $\lambda = \ln(1 - \frac{2m}{r})$, the differential equation for k_g can be integrated and the spacetime metric reads

$$ds^2 = e^{2(k_s+k_e)} g^\gamma \frac{dr^2}{1 - 2m/r} + g^\gamma r^2 (e^{2(k_s+k_e)} d\theta^2 + \sin^2 \theta d\varphi^2) - \frac{(1 - 2m/r)}{g^\gamma} dt^2 \quad (56)$$

the scalar and the electromagnetic fields for this solution read

$$\begin{aligned}\kappa^2 &= e^{-2\alpha\phi} = \frac{e^{-2\alpha\phi_0}}{(1 - 2m/r)g^\beta} \\ A_{3,\rho} &= -Q\rho\tau_{,z}, \quad A_{3,z} = Q\rho\tau_{,\rho}\end{aligned}\tag{57}$$

Metric (56), (57) is an static asymptotically flat metric with mass parameter m , and magnetostatic charge Q . The scalar parameter depends on the form of τ . Observe that the electromagnetic field is always integrable because τ fulfills the Laplace equation $D(\rho D\tau) = (\rho\tau_{,z})_{,z} + (\rho\tau_{,\rho})_{,\rho} = 0$. So we can construct solutions representing magnetic monopoles, dipoles, etc. With this form of the metric we can interpret g and e^{2k_e} as the contribution of the electromagnetic field and e^{2k_s} as the contribution of the scalar field to the metric. Asymptotically, the scalar field behaves like $e^{-2\alpha\phi_0}/g^\beta(1 + 2m/r + \dots)$, where $g \sim 1 + O(1/r^n)$. This means that the scalar field deviate from a constant in orders of $2m/r$. For a star like the sun $2m/r \sim 10^{-6}$, for a white dwarf $2m/r \sim 10^{-4}$ and only for a compact star like a neutron star $2m/r \sim 10^{-1}$, *i.e.*, this metric represents the space-time of an object very similar as a magnetized Schwarzschild one. Only for a very compact object both metrics are different. This fact is in agreement with the concept of spontaneous scalarization: if scalar fields exist, compact stars will prefer to posses one in order to save energy, but even when a star posses one, it will very difficult to detect it (see [8]). We can carry out the follow classification of solution (56), (57):

1. Suppose that $\tau = 0$, this implies that $g = 1$ and $A_3 = 0$, so that the metric (56) is

$$ds^2 = e^{2k_s} \frac{dr^2}{1 - 2m/r} + r^2(e^{2k_s} d\theta^2 + \sin^2 \theta d\varphi^2) - (1 - 2m/r) dt^2 \tag{58}$$

This metric is almost spherically symmetric and represents a gravitational body (gravitational monopole) with scalar field. The scalar field deforms the spherical symmetry in the factor $e^{2k_s} d\theta^2$. If $k_s = 0$ we recover the spherical symmetry. This metric has been used as a model for a star [5], finding that the physical differences between (58) and the Schwarzschild solution are too small to be measured, even for a compact star like a pulsar. This implies that a star like the sun could posses a scalar field and we will not be able to measure it.

2. We now take $\tau \neq 0$, $g = 1$, but with arbitrary magnetic field A_3 . The metric (56) then reads

$$ds^2 = e^{2(k_s+k_e)} \frac{dr^2}{1-2m/r} + r^2(e^{2(k_s+k_e)}d\theta^2 + \sin^2\theta d\varphi^2) - (1-2m/r)dt^2 \quad (59)$$

In this case the scalar and the magnetostatic potentials are the ones that deform the spherical symmetry of the metric. If we furthermore make $k_s + k_e = 0$ and $\tau = \lambda$, $(p+q-1)^2 - 2pq\beta = 0$, we recover the spherical symmetry and the metric transforms into the Schwarzschild line element. This metric represents a gravitational body with arbitrary magnetostatic field coupled to a scalar field. The scalar and mass parameters here are proportional.

3. If we choose $\tau = \lambda$, metric (56) is spherically symmetric, the constants fulfill the constrain $(p+q-1)^2 - pq\beta = 0$, and the metric (56) reads

$$ds^2 = g^\gamma \frac{dr^2}{1-2m/r} + g^\gamma r^2(d\theta^2 + \sin^2\theta d\varphi^2) - \frac{(1-2m/r)}{g^\gamma} dt^2 \quad (60)$$

Here is contained the Gibbons- Maeda black hole [19] by choosing τ to be the harmonic function corresponding to a monopole. The generalized version of solution [19] for any electromagnetic multipole field is given by this metric.

4. Finally, we choose $\tau = \lambda$ and $p+q = 1$, which implies $k_s = 0$. The metric (56) is now given by

$$ds^2 = e^{2k_e} g^\gamma \frac{dr^2}{1-2m/r} + g^\gamma r^2(e^{2k_e}d\theta^2 + \sin^2\theta d\varphi^2) - \frac{(1-2m/r)}{g^\gamma} dt^2 \quad (61)$$

This metric is again almost spherically symmetric, but now it is deformed by the electromagnetic field, the factor $e^{2k_e}d\theta^2$ is the deformation of the spherical symmetry due to the electromagnetic field.

4.3 Third Class

Now let us study the case when the electromagnetic field vanishes. Here it is convenient to take the harmonic parameter complex. Choosing the simple

ansatz $a_2 = b_1 = b_2 = c_1 = c_2 = 0$, $\sigma = 1$, using Eq.(24) with the complex parameter ξ , and repeating the procedure, we obtain that the system of equations Eqs. (26), (27) and (28) takes the form

$$\begin{aligned} a_{1,\xi} - a_1^2 - \frac{2\xi^* a_1}{1 - \xi\xi^*} &= 0, \\ a_{1,\xi^*} + a_1 a_1^* &= 0, \end{aligned}$$

with solution

$$a_1 = \frac{1 + \xi^*}{(1 + \xi)(\xi\xi^* - 1)}. \quad (62)$$

Performing the integration to obtain the potential functions, one finally gets

$$\begin{aligned} f &= \frac{\xi\xi^* - 1}{(1 + \xi)(1 + \xi^*)}, \\ \epsilon &= i \frac{\xi - \xi^*}{(1 + \xi)(1 + \xi^*)}, \end{aligned} \quad (63)$$

which we identify with the Ernst solution, that is, we have obtained the Tomimatsu-Sato family of solutions, including of course Kerr as a particular case [12].

4.4 Fourth Class

Finally, we obtain some stationary solutions using this method. The most simple ansatz to solve equations (26), (27) and (28) is supposing that all the functions a_1, a_2, \dots etc. are constants, the differential equations become then algebraic equations which can be easily solved. For this case we take the ansatz

$$\begin{aligned} a_1 &= ip, \quad a_2 = 0, \\ b_1 &= p(1 - i), \quad b_2 = 0, \\ d_1 &= -2p, \quad c_2 = 0, \end{aligned} \quad (64)$$

where p is an arbitrary constant. Integrating the potentials for this case we obtain

$$f = 1,$$

$$\begin{aligned}
\kappa^2 &= \kappa_0^2 e^{-4p\lambda}, \\
\psi &= -\frac{e^{2p\lambda}}{\kappa_0} + \psi_0, \\
\chi &= \kappa_0 e^{-2p\lambda} + \chi_0, \\
\epsilon &= \kappa_0 \psi_0 e^{-2p\lambda}.
\end{aligned} \tag{65}$$

Substituting in the metric (2) and in the equation for the function k , (19), the metric can be integrated to obtain

$$ds^2 = -(dt - \omega d\varphi)^2 + [e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \tag{66}$$

where ω and the function k can be integrated from the differential equations:

$$\begin{aligned}
\omega_{,\rho} &= -2p \rho \lambda_{,z} \quad \omega_{,z} = 2p \rho \lambda_{,\rho} \\
k_{,\zeta} &= \frac{p^2(3\alpha^2 + 4)}{\alpha^2} \rho(\lambda_{,\zeta})^2
\end{aligned} \tag{67}$$

The integrability conditions for ω and k are guaranteed because λ fulfills the Laplace equation. Metric (67) represents a rotating degenerated object (the gravitational potential $f = 1$), with electromagnetic and scalar potentials, where the coupling constant α between scalar and electromagnetic fields remains arbitrary. The particular multipole development depends on the harmonic function λ . As an example, if $\lambda = \cos\theta/r$, the solution represents a pure magnetic monopole with a multipole electrostatic field, without gravitational one. In this case $\omega = -2p \sin^2\theta/r$, and the magnetic field is proportional to it. This represents thus a magnetic dipole, with an exponentially decaying scalar field.

5 Conclusions

We have presented a detailed description of the functional space formulation, joined with the harmonic maps one, in such a way that starting with the Einstein-Hilbert action for the stationary axisymmetric space-time, we obtained an effective Lagrangian for the field variables, by means of a Legendre transformation obtained a ‘‘Hamiltonian’’ and with a canonical transformation reduce the system of dynamical equations to three first order coupled differential equations for three unknowns, A, B, C , two complex and

one real. The harmonic maps ansatz enabled us to reduce the system of five second order partial differential equations (12) for the Einstein-Maxwell-Dilaton system with two Killing vectors to a system of five (two complex and one real) first order ordinary differential equations (22). Using the harmonic maps ansatz [9], we rewrote that system of equations in such a way that, for different ansatz, it allows to generate large classes of solutions to the Einstein-Maxwell theory non minimally coupled to a dilatonic field. In this way, we consider that we have described a robust formulation which allows us to generate large classes of solutions, placing us on the right track, we think, to obtain exact solutions for astrophysical important cases, like the one describing a compact charged rotating object surrounded by a scalar field which could be useful for solving the dark matter problem at a galactic level. Actually, in [1, 26], we have worked out this idea and the scalar field does is a good candidate for the dark matter in the galactic halo of spiral galaxies. We have presented several particular cases which include most of the well known solutions, as well as new ones, where we have explicitly presented the form of the fields and of the charges, for particular choices of the potentials. Also, the formulation described in the present work is not only a technique to generate exact solutions, but also it has several other possible directions worth studying, for instance, from the effective Lagrangian (6) an off-shell Lagrangian for the non-linear σ -model for some families of solutions can be obtained [25]. It has not been possible to make this analogy for cases with different values of α , due to the fact that for them the potential space is not symmetric. Finally we think that the “Hamiltonian” obtained in the present formulation, is worth to study further within the gravity quantization endeavor.

6 Acknowledgments

This work has been supported by DGAPA-UNAM, project IN121298, and by CONACYT, Mexico, project 3697-E. TM thanks the hospitality from the relativity group in Jena, Germany, and the DAAD support while this work was partially done.

References

- [1] T. Matos and F. S. Guzmán, *Class. Quant. Grav.*, **17**, (2000), L9-L16. Available at: *gr-qc/9810028*. T. Matos and F. S. Guzmán. *Ann. Phys. (Leipzig)* **9**, (2000), S133-S136. Available at: *astro-ph/0002126*
- [2] F. S. Guzmán, T. Matos and H. Villegas, *astro-ph/9811143*, to be published.
- [3] T. Damour and G. Esposito-Farese. *Phys. Rev. Lett.* **70**, (1993), 2220.
- [4] T. Damour and G. Esposito-Farese. *Class. Quant. Grav.*, **9**, (1992), 2093.
- [5] Tonatiuh Matos, Octavio Obregon and Hugo Villegas. *Mod. Phys. Lett. A***13**, (1998), 3161. T. Matos and H. Villegas, *Astro. Phys. J.*, (1999), to be published.
- [6] G. Esposito-Farese, *gr-qc/9612039*.
- [7] T. Matos, D. Nuñez and H. Quevedo. *Phys. Rev.* **D51**, (1995), R310.
- [8] T. Matos. *Gen. Rel. Grav.*, **30**, (1998), 5.
- [9] T. Matos. *J. Math. Phys.* **35**, (1994), 1302.
- [10] T. Matos. ‘*Exact Solutions of G-invariant Chiral Equations*’ *Math. Notes*, **58**, (1995), 1178-1182.
- [11] G. Neugebauer. *Habilitationsschrift* (Universität Jena Press, Jena, Germany) (1969).
- [12] D. Kramer, H. Stephani, M. MacCallum and E. Held. *Exact Solutions of Einstein Field Equations*. (1980), DVW, Berlin.
- [13] T. Matos. *Astron. Nachr.* **307**, (1986), 317.
- [14] G. Neugebauer and D. Kramer. *Ann. Phys. (Leipzig)*, **24**, (1969), 62.
- [15] Tonatiuh Matos, Guadalupe Rodríguez and Ricardo Becerril. *J. Math. Phys.* **33**, (1992), 3521.

- [16] Misner Ch. W., Thorne K. S., and Wheeler, J. A. (1973) *Gravitation* (Freeman and Co., San Francisco) Ch. 21.
- [17] D. Núñez, H. Quevedo and A. Sánchez. *Rev. Mex. de Fis.*, **44**, (1998), 1820.
- [18] D. Kastor and J. Traschen, *Phys. Rev.* **D47**, 5370, (1993); J. H. Horne and G. T. Horowitz, *ibid*, **48**, R5457 (1993).
- [19] G. Gibbons and K. Maeda. *Nuc. Phys.* **B298**, 741 (1988). D. Garfinkle, G. T. Horowitz and A. Strominger. *Phys. Rev.* **D43**, 3140, (1991).
- [20] T. Matos. *Phys. Rev.* **D49**, 4296, (1994).
- [21] M. Rios. *Master thesis*, CINVESTAV, (1995).
- [22] T. Matos and A. Macias. *Mod. Phys. Lett.* **A9**, 3707, (1994).
- [23] A. Macias and T. Matos. *Class. Quant. Grav.* **13** , (1996), 345.
- [24] T. Matos and R. Becerril, *Rev. Mex. Fis.* **38**, 69, (1992).
- [25] J. Cortez, D. Nuñez and H. Quevedo. *Int. J. of Modern Phys.*, in press.
- [26] T. Matos, F. S. Guzmán and D. Nuñez. *Phys Rev.* **D57**, (2000), in press. Available at: *astro-ph/0003398*. T. Matos, D. Nuñez, F. Guzman, E. Ramirez, sent to *Astrophys. J.*;